平面空間 $\xrightarrow{\text{平面變換}}$ 平面空間

Linear Transformation

$\begin{cases} V \\ W \end{cases}$ vector space 基底 $\{v_1, \cdots, v_n\}$ $\dim V = n$, $\{w_1, \cdots, w_m\}$ $\dim W = m$

- $T: V \longrightarrow W$ is linear $\overset{def}{\Longleftrightarrow}$ $\forall \begin{matrix} u, v \in V \\ a, b \in \mathbb{C} \end{matrix}$ $\begin{cases} \text{(i) } T(u+v) = T(u) + T(v) \\ \text{(ii) } T(au) = aT(u) \end{cases}$

  (homomorphism 同態)

  $\Longleftrightarrow$ $T(au+bv) = aT(u) + bT(v)$

- $L(V,W) = \mathrm{Hom}(V,W) := \{T \mid T: V \to W \text{ linear}\}$

  $\left[T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3) \right.$ (linear property)

- 定理 (1) $T(0_V) = T(0_W)$

  (2) $L(V,W)$: vector space $\begin{cases} T_1 + T_2 \\ aT \end{cases}$ (函數定義)

  (3) $\dim L(V,W) = nm$, basis $= \{T_{ij}\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ $T_{ij}: \begin{cases} v_j \to w_i \\ v_k \to 0 \quad (k \neq j) \end{cases}$ $(|i\rangle\langle j|)$

  (4) $\mathrm{Im}(T) = \{T(x) \mid x \in A\} \lhd W$

  $\mathrm{Ker}(T) = \{x \in V \mid T(x) = 0\} \lhd V$

  $\mathrm{Ker}(T) = \{0\} \Leftrightarrow T: 1\text{-}1$ (T: isomorphism)

  ($V \cong W$ isomorphic 同構)

  (5) $\dim V = \dim W = n \Rightarrow V \cong W$

  証 $v_i \longmapsto w_i$ $(1 \le i \le n)$

  $\sum_{i=1}^{n} a_i v_i \to \sum_{i=1}^{n} a_i w_i$ (isomorphism)

  推論: $H_n \cong \mathbb{C}^n$

  $a_0|0\rangle + \cdots + a_{n-1}|n-1\rangle \mapsto a_0 e_1 + \cdots + a_{n-1} e_n = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$

  (6) $L(V,W) \cong M_{m \times n}$ $T_{ij} \to M_{ij} = \begin{array}{c} j \\ \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \leftarrow i \end{array}$

  $\begin{cases} T \longrightarrow [T] = A = [T(e_1) \; T(e_2) \cdots T(e_n)]_{m \times n} \\ T_A \longleftarrow A \\ T_A(x) = Ax \end{cases}$

  $\Rightarrow T(x) = T\left(\sum_i x_i e_i\right) = \sum_i x_i T(e_i) = \sum_i x_i a_i = Ax$

例 $T: \mathbb{C}^4 \longrightarrow \mathbb{C}^3$

(甲) $T(x) = T\left(\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}\right) = T(xe_1 + ye_2 + ze_3 + ue_4)$

$= xT(e_1) + yT(e_2) + zT(e_3) + uT(e_4)$

(乙) $Ax = [a_1 \; a_2 \; a_3 \; a_4] \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = xa_1 + ya_2 + za_3 + ua_3$

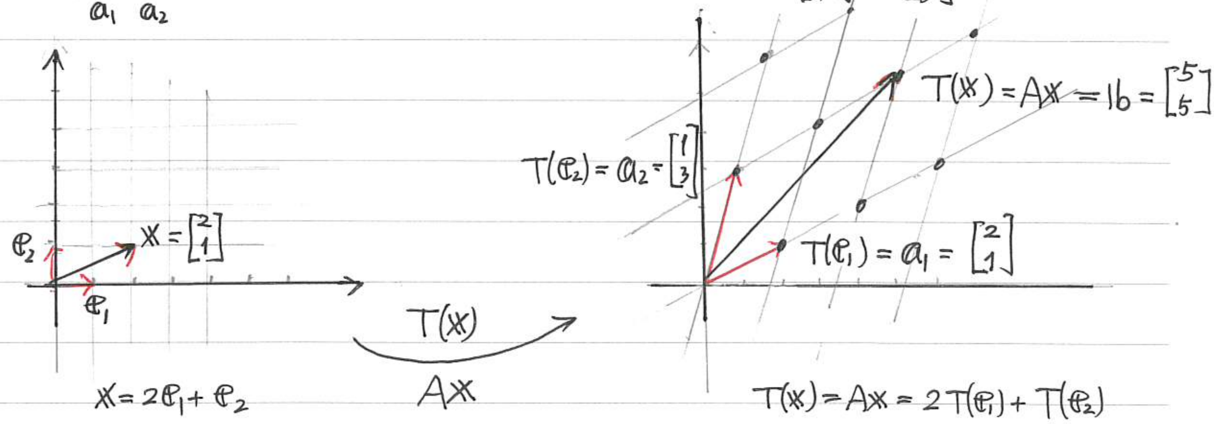
$$\underbrace{T\left(\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}\right) = \begin{bmatrix} x - y + z + 3u \\ 2x - 3y - z + u \\ 3x + y + 4z + u \end{bmatrix}}_{T(x)} = \underbrace{\begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -3 & -1 & 1 \\ 3 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}}_{Ax}$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 2x+y \\ x+3y \end{bmatrix}$$

$$\begin{cases} T(\mathbb{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \leftarrow a_1 \\ T(\mathbb{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \leftarrow a_2 \end{cases}$$

$$\begin{cases} T_A(\mathbb{x}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T(x\mathbb{e}_1 + y\mathbb{e}_2) = xT(\mathbb{e}_1) + yT(\mathbb{e}_2) = x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ A\mathbb{x} = \begin{array}{c} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \\ a_1 \ \ a_2 \end{array} \begin{bmatrix} x \\ y \end{bmatrix} = x\, a_1 + y\, a_2 = x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{cases} = \begin{bmatrix} 2x+y \\ x+3y \end{bmatrix} = \mathbb{b}$$



regular (正則)

Theorem (nonsingular matrix) $A_{n\times n} : \mathbb{R}^n \to \mathbb{R}^n$ $\leftrightarrow$ Singular

下列敘述等價，皆可作為 nonsingular matrix 的定義

(1) $\operatorname{rank}(A) = \#\text{pivots} = n$

(2) $A\mathbb{x} = \mathbb{b}$ 有唯一解 $(A^{-1}\mathbb{b})$

(3) $A\mathbb{x} = \mathbb{0}$ 只有 0 解

(4) A^{-1} 存在

(5) $\det(A) \neq 0$

(6) $\{A_1, \dots, A_n\}$ basis for $\mathbb{R}^n$

(7) $\{a_1, \dots, a_n\}$ 〃

(8) $N(A) = \operatorname{Ker}(T_A) = \{\mathbb{0}\}$ $\Leftrightarrow$ T_A : 1-1

(9) $\operatorname{Im}(A) = \operatorname{Im}(T_A) = \mathbb{R}^n$ $\Leftrightarrow$ T_A onto

(10) 0 不是 eigen value

Change of basis (座標變換)

$P\mathbb{x}' = x_1' v_1 + \cdots + x_n' v_n = \mathbb{x}$

定理: $T_A : V \to V$ $\quad \begin{cases} \mathcal{B} = \{e_1, \dots, e_n\} & A \\ \mathcal{B}' = \{v_1, \dots, v_n\} & A' \end{cases}$

$P = [v_1, \dots, v_n]$ 變換矩陣

$$\begin{tikzcd} \mathbb{x} \arrow[d, "A"'] & \mathbb{x}' \arrow[l, "P"'] \arrow[d, "A'"] \\ A\mathbb{x} \arrow[r, "P^{-1}"] & A'\mathbb{x}' \end{tikzcd}$$

原座標 $\mathcal{B}$ $\qquad$ 新座標 $\mathcal{B}'$

$\Rightarrow A' = P^{-1}AP$

例 $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ $\qquad \mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

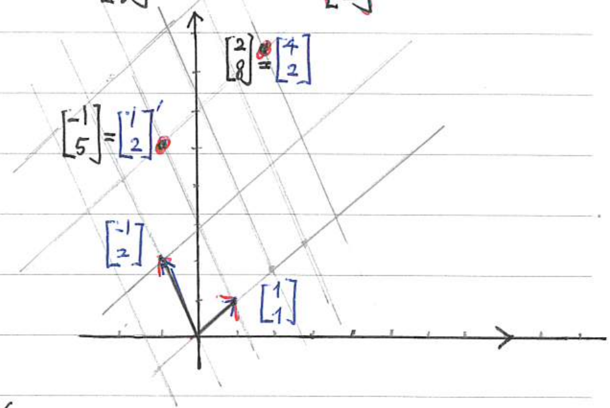
$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

$$\begin{tikzcd} \begin{bmatrix} -1 \\ 5 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \arrow[l, "\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}"'] \arrow[d, "A' = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}"] \\ \begin{bmatrix} 2 \\ 8 \end{bmatrix} \arrow[u, leftarrow, "A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}"] \arrow[r, "\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}"] & \begin{bmatrix} 4 \\ 2 \end{bmatrix} \end{tikzcd}$$

$$A' = P^{-1} \, A \, P$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 8 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



定理 (矩陣對角化): $A_{n \times n}$

$\exists\ \mathcal{B}' = \{v_1, \dots, v_n\}$ eigen vectors $\quad \begin{cases} Av_1 = \lambda_1 v_1 \\ \vdots \\ Av_n = \lambda_n v_n \end{cases}$

$\lambda_1, \dots, \lambda_n$ 〃 values

$$\Rightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} = \Lambda, \quad P = [v_1, \dots, v_n]$$

Dual Space

V : vector space $\begin{cases} \dim V = n \\ \text{basis} = \{v_1, \cdots, v_n\} \end{cases}$

$\hat{V} = L(V, \mathbb{C})$: dual space of V

Ψ

$f : V \to \mathbb{C}$ linear functional (泛函), $f(ax+by) = af(x) + bf(y) \in \mathbb{C}$

性質

(1) $\dim \hat{V} = n$, dual basis $= \{\hat{v}_1, \cdots, \hat{v}_n\}$, $\hat{v}_j(v_k) = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$

$f = a_1\hat{v}_1 + \cdots + a_n v_n$, $a_j = f(v_j)$ $(\langle j|)$

Example $V = \mathbb{C}^3$ $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\hat{V}$: basis $= \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ $\hat{e}_j(e_k) = \delta_{jk}$

Ψ $\cong \{[1\ 0\ 0], [0\ 1\ 0]\ [0\ 0\ 1]\}$

$f = [a \quad b \quad c]$ $a = f(e_1)$, $b = f(e_2)$, $c = f(e_3)$

$\cong a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3$

檢驗 (i) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = f(xe_1 + ye_2 + ze_3)$

$= xf(e_1) + yf(e_2) + zf(e_3)$

$= xa + yb + zc = [a \quad b \quad c]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(ii) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = (a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3)(xe_1 + ye_2 + ze_3)$

$= ax + by + cz$

(2) $V \cong \hat{V}$

Given $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in V$, $f_v \overset{?}{=} [a\ b\ c] \in \hat{V}$, (dual of v)

$= a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3$ $\begin{cases} a = \hat{e}_1(v) \\ b = \hat{e}_2(v) \\ c = \hat{e}_3(v) \end{cases}$

$f_v(x) = ax + by + cz$

$= [a\ b\ c]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Inner product Space (内積空間)

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$(V, \langle \cdot,\cdot \rangle)$ 內積空間 $\forall u, v \in V$, $a, b \in \mathbb{C}$

(0) $\langle u,v \rangle \in \mathbb{C}$
(1) $\langle u,v \rangle = \overline{\langle v,u \rangle}$
(2) $\langle u,u \rangle \geq 0$, 且 $\langle u,u \rangle = 0$ iff $u = 0$
(3) $\langle u, av+bw \rangle = a\langle u,v \rangle + b\langle u,w \rangle$ (右線性)

定義

(1) $\|u\| = \sqrt{\langle u,u \rangle}$ (norm) (u: normal if $\|u\| = 1$)
(2) $u \perp v$ iff $\langle u,v \rangle = 0$ (orthogonal)
(3) $\{v_1, \dots, v_k\}$ orthonormal if $\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$
(4) V is Hilbert Space if V: 有限維內積空間
(5) $W \triangleleft V$, $W^{\perp} := \{ v \in V \mid v \perp w, \forall w \in W \}$ (orthogonal Complement)

例

(1) $\mathbb{C}^n$, $u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $v = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, $\langle u,v \rangle = u^* v = \bar{a}_1 b_1 + \bar{a}_2 b_2 + \cdots + \bar{a}_n b_n$

(2) $\mathcal{H}_n$ $\quad |u\rangle = a_0|0\rangle + \cdots + a_{n-1}|n-1\rangle$, $|v\rangle = b_0|0\rangle + \cdots + b_{n-1}|n-1\rangle$ $\quad \langle |u\rangle, |v\rangle \rangle = \langle u|v \rangle = \bar{a}_0 b_0 + \cdots + \overline{a_{n-1}} b_{n-1}$

定理:

(1) $\langle u,u \rangle \in \mathbb{R}$ $\quad \because \langle u,u \rangle = \overline{\langle u,u \rangle}$
(2) $\langle au+bv, w \rangle = \bar{a}\langle u,w \rangle + \bar{b}\langle v,w \rangle$
(3) $\|au\| = |a|\,\|u\|$
(4) $|\langle u,v \rangle| \leq \|u\|\,\|v\|$ (Schwartz 不等式)
(5) $\|u+v\| \leq \|u\| + \|v\|$ (三角 〃)
(6) $W^{\perp} \triangleleft V$
(7) $\{v_1, \dots, v_k\}$ orthonormal $\Rightarrow \begin{cases} \text{(i) } \{v_i\} \text{ independent} \\ \text{(ii) } w = a_1 v_1 + \cdots + a_k v_k \Rightarrow a_i = \langle v_i, w \rangle \end{cases}$
(8) Hilbert space 存在 orthonormal basis (Gram-schmidt)
(9) $\langle u, Av \rangle = \langle A^* u, v \rangle$ $\quad \because R = (A^* u)^* v = u^* A v = L$
(cross the product)

Adjoint $A^* = \overline{A^T}$, $\langle x, Ay\rangle = \langle A^*x, y\rangle$

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定理 (1)(2)(3) 任兩個成立 ⇒ 第三個成立，$A_{n\times n}$ | $\mathbb{R}^n$

(1) unitary : $A^*A = I$ $(\Rightarrow AA^* = I)$ | $A^TA = I$ orthogonal

(2) involution : $AA = I$

(3) Hermitian : $A^* = A$ | $A^T = A$ symmetric

定理 $U_{n\times n} = [u_1, \cdots, u_n]$ 下列 (1)~(4) 等價

(1) $U^*U = I$ (unitary)

(2) $\{u_1, \cdots, u_n\}$ orthonormal (各列/行 orthonormal)

(3) $\langle Ux, Uy\rangle = \langle x, y\rangle$ (保積 preserve inner product) $\}$ ⇒ 保角

(4) $\|Ux\| = \|x\|$ (〃 長 〃 length)
$\langle Ux, Ux\rangle = \langle x, x\rangle$

証
$$\begin{array}{ccc} (1) & \Longleftrightarrow & (2) \\ \Downarrow & \nearrow & \\ (3) & \Longleftrightarrow & (4) \end{array}$$

(1) ⇒ (3) $\langle Ux, Uy\rangle = \langle U^*Ux, y\rangle = \langle x, y\rangle$

(3) ⇒ (2) $\langle u_i, u_j\rangle = \langle Ue_i, Ue_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$

(4) ⇒ (3) (甲) $\langle U(x+y), U(x+y)\rangle = \langle x+y, x+y\rangle$

$\Rightarrow \langle Ux+Uy, Ux+Uy\rangle = \langle x+y, x+y\rangle$

$\Rightarrow \langle Ux, Uy\rangle + \langle Uy, Ux\rangle = \langle x, y\rangle + \langle y, x\rangle$

(乙) $\langle U(x+iy), U(x+iy)\rangle = \langle x+iy, x+iy\rangle$

$\Rightarrow i\langle Ux, Uy\rangle - i\langle Uy, Ux\rangle = i\langle x, y\rangle - i\langle y, x\rangle$

定理 (Spectral th) $A^* = A$

(1) all eigen values $\in \mathbb{R}$ $[\ \bar{\lambda}\langle x, x\rangle = \langle Ax, x\rangle = \lambda\langle x, x\rangle \quad \therefore \bar{\lambda} = \lambda\ ]$

(2) $\begin{cases} Ax = \lambda x \\ Ay = \mu y \end{cases}$, $\lambda \neq \mu \Rightarrow x \perp y$, $[\ \bar{\lambda}\langle x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \mu\langle x, y\rangle$
$\therefore \langle x, y\rangle = 0\ ]$

(3) $\exists$ orthonormal eigen basis $\{u_1, \cdots, u_n\}$

(4) $A = P\Lambda P^* = [u_1, \cdots, u_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\begin{bmatrix}u_1^* \\ \vdots \\ u_n^*\end{bmatrix}$

$= \lambda_1 u_1 u_1^* + \lambda_2 u_2 u_2^* + \cdots + \lambda_n u_n u_n^*$